

A Posteriori Error Estimation for Highly Indefinite Helmholtz Problems

Willy Dörfler · Stefan Sauter

Abstract — We develop a new analysis for residual-type a posteriori error estimation for a class of highly indefinite elliptic boundary value problems by considering the Helmholtz equation at high wavenumber $k > 0$ as our model problem. We employ a classical conforming Galerkin discretization by using hp -finite elements. In [Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions, Math. Comp., 79 (2010), pp. 1871–1914], Melenk and Sauter introduced an hp -finite element discretization which leads to a stable and pollution-free discretization of the Helmholtz equation under a mild *resolution condition* which requires only $O(k^d)$ degrees of freedom, where $d = 1, 2, 3$ denotes the spatial dimension. In the present paper, we will introduce an a posteriori error estimator for this problem and prove its reliability and efficiency. The constants in these estimates become *independent* of the, possibly, high wavenumber $k > 0$ provided the aforementioned resolution condition for stability is satisfied. We emphasize that, by using the classical theory, the constants in the a posteriori estimates would be amplified by a factor k .

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1. Introduction

In this paper, we will introduce a new analysis for residual-based a posteriori error estimation. We consider the conforming Galerkin method with hp -finite elements applied to a class of highly indefinite boundary value problems, which arise, e.g., when electromagnetic or acoustic scattering problems are modelled in the frequency domain. As our model problem we consider a highly indefinite Helmholtz equation with oscillatory solutions.

Residual-based a posteriori error estimates for elliptic problems have been introduced in [4, 5] and their theory for elliptic problems is now fairly completely established (cf. [1, 19]). To sketch the principal idea and to explain the goal of this paper let u denote the (unknown) solution of the weak formulation of an elliptic second order PDE with appropriate boundary conditions. Typically the solution belongs to some infinite-dimensional Sobolev space H . Let u_S denote a computed Galerkin solution based on a finite dimensional subspace $S \subset H$.

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A (reliable) a posteriori error estimator is a *computable* expression η which depends on u_S and the given data such that an estimate of the form

$$\|u - u_S\|_H \leq C\eta(u_S) \quad (1.1)$$

holds for a constant C which either is known explicitly or sharp upper bounds are available. We emphasize that in the literature various refinements of this concept of a posteriori error estimation exist while for the purpose of our introduction this simple definition is sufficient.

In the classical theory the constant C depends linearly on the norm of the solution operator of the PDE in some appropriate function spaces, more precisely, it depends reciprocally on the *inf-sup constant* γ (cf. [5, Theorem 3.2]). In [12] it was proved for the Helmholtz problem with Robin boundary conditions that for certain classes of physical domains the reciprocal inf-sup constant $1/\gamma$ (and, hence, also the constant C in (1.1)) grows linearly with the wavenumber. See also [9] for further estimates of the inf-sup constant for the Helmholtz problem. However, this implies that for large wavenumbers the classical a posteriori estimation becomes useless because the error then typically is highly overestimated. Additional difficulties arise for the a posteriori error estimation for highly indefinite problems because the existence and uniqueness of the classical Galerkin solution is ensured only if the mesh width is sufficiently small.

In contrast to definite elliptic problems, there exist only relatively few publications in the literature on a posteriori estimation for highly indefinite problems (cf. [2, 3, 10, 18]).

In [14] and [15] a new a priori convergence theory for Galerkin discretizations of highly indefinite boundary value problems has been developed which is based on new regularity estimates (the *splitting lemmas* as in [14] and [15]), where the solution $u = u_{\text{rough}} + u_{\text{osc}}$ is split into a “rough part” and a “smooth part” with high-order regularity in (weighted) Sobolev spaces. Under appropriate assumptions on the smoothness of the data (domain Ω , right-hand side f) the following regularity estimates hold:

$$\|u_{\text{rough}}\|_{H^2(\Omega)} \leq C_f \quad \text{and} \quad \|u_{\text{osc}}\|_{H^m(\Omega)} \leq C_f k^{m-1},$$

where the constant C_f is independent of k . This theory allows in the a priori convergence theory to “absorb” the k -depending L^2 -error of the PDE into the wavenumber-independent part of the equation. As a consequence, discrete stability and pollution-free a priori convergence estimates can be proved under a very mild *resolution condition* on the hp -finite element space which requires only $O(k^d)$ degrees of freedom.

In this paper, we will develop a new a posteriori analysis based on similar ideas: The L^2 -part of the a posteriori error will be estimated by the H^1 -error and then can be compensated by an appropriate choice of the hp -finite element space. This allows to prove reliability and efficiency of the error estimator. The constants in these estimates become *independent* of the, possibly, high wavenumber $k > 0$ provided the aforementioned resolution condition for stability is satisfied. We emphasize that, by using the classical theory, the constants in the a posteriori estimates would be amplified by a factor k .

The paper is structured as follows. In Section 2, we will consider as our model problem the high frequency, time harmonic scattering of an acoustic wave at some bounded domain in an unbounded exterior domain and transform it to a finite domain by using a Dirichlet-to-Neumann boundary operator or some approximation to it. We define a conforming Galerkin hp -finite element discretization for its numerical approximation.

In Section 3, we summarize the a priori analysis as in [14] and [15] which will be needed to determine the minimal hp -finite element space for a stable Galerkin discretization.

In Section 4, we will present the a posteriori error estimator and prove its reliability and efficiency. It will turn out that the optimal polynomial degree p will depend logarithmically on the wavenumber and, hence, the finite element interpolation theory has to be explicit with respect to the mesh width h and the polynomial degree p .

In a forthcoming paper, we will focus on numerical experiments and also on the definition of an hp -refinement strategy in order to obtain a convergent adaptive algorithm.

2. Model Helmholtz Problems and their Discretization

2.1. Model Problems

The Helmholtz equation describes wave phenomena in the frequency domain which, e.g., arises if electromagnetic or acoustic waves are scattered from or emitted by bounded physical objects. In this light, the computational domain for such wave problems, typically, is the unbounded complement of a bounded domain $\Omega^{\text{in}} \subset \mathbb{R}^d$, $d = 1, 2, 3$, i.e., $\Omega^{\text{out}} := \mathbb{R}^d \setminus \overline{\Omega^{\text{in}}}$. Throughout this paper, we assume that Ω^{in} has a Lipschitz boundary $\Gamma^{\text{in}} := \partial\Omega^{\text{in}}$.

The Helmholtz problem depends on the wavenumber k . In most parts of the paper (exceptions: Remarks 3.5, 4.2 and Corollaries 4.10, 4.11) we allow for variable wavenumber $k : \Omega^{\text{out}} \rightarrow \mathbb{R}$ but always assume that k is real-valued, nonnegative, and a positive constant outside a sufficiently large ball (cf. (2.7)).

For a given right-hand side $f \in L^2(\Omega^{\text{out}})$, the *Helmholtz problem* is to seek $U \in H_{\text{loc}}^1(\Omega^{\text{out}})$ such that

$$(-\Delta - k^2)U = f \quad \text{in } \Omega^{\text{out}} \quad (2.1a)$$

is satisfied. Towards infinity, *Sommerfeld's radiation condition* is imposed:

$$|\partial_r U - i k U| = o(|x|^{\frac{1-d}{2}}) \quad \text{for } |x| \rightarrow \infty, \quad (2.1b)$$

where ∂_r denotes differentiation in radial direction and $|\cdot|$ the Euclidian vector norm. For simplicity we restrict here to the *homogeneous Dirichlet boundary condition* on Γ^{in} :

$$U|_{\Gamma^{\text{in}}} = 0. \quad (2.1c)$$

We assume that f is local in the sense that there exists some bounded, simply connected Lipschitz domain¹ Ω^* such that a) $\overline{\Omega^{\text{in}}} \subset \Omega^*$, b) $\text{supp}(f) \subset \Omega^*$, and c) k is constant in a neighborhood of $\partial\Omega^*$ and anywhere outside Ω^* . The *computational domain* (cf. Figure 1) will be

$$\Omega := \Omega^* \setminus \overline{\Omega^{\text{in}}} \quad \text{with boundary } \Gamma^{\text{in}} \cup \Gamma^{\text{out}} \text{ and } \Gamma^{\text{out}} := \partial\Omega^*.$$

It is well known that problem (2.1) can be reformulated as: Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} (-\Delta - k^2)u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma^{\text{in}}, \\ \partial_n u &= T_k u && \text{on } \Gamma^{\text{out}}, \end{aligned} \quad (2.2)$$

¹Since Ω^{in} is bounded, Ω^* can always be chosen as a ball. Other choices of Ω^* might be preferable in certain situations.

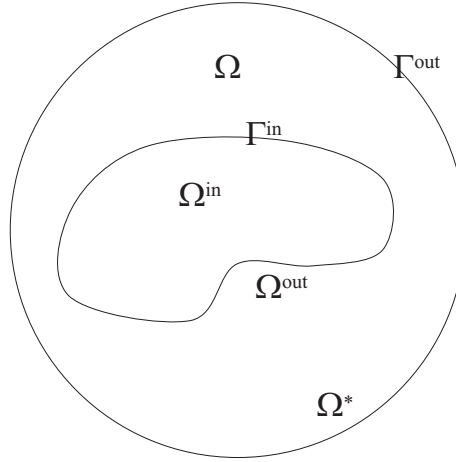


Figure 1. Scatterer Ω^{in} with boundary Γ^{in} and exterior domain Ω^{out} . The support of f is assumed to be contained in the bounded region Ω^* . The domain for the weak variational formulation is $\Omega = \Omega^* \setminus \Omega^{\text{in}}$.

where T_k denotes the Dirichlet-to-Neumann operator for the Helmholtz problem² and ∂_n is the normal derivative at Γ^{out} . The previous problems are posed in the *weak formulation* given by: Find

$$u \in \mathcal{H} := \{u \in H^1(\Omega) \mid u|_{\Gamma^{\text{in}}} = 0\} \quad (2.3)$$

such that

$$A_{\text{DtN}}(u, v) := \int_{\Omega} (\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v}) - \int_{\Gamma^{\text{out}}} (T_k u) \bar{v} = \int_{\Omega} f \bar{v} \quad \text{for all } v \in \mathcal{H}. \quad (2.4)$$

Since the numerical realization of the nonlocal DtN map T_k is costly, various approaches exist in the literature to approximate this operator by a local operator. The most simple one is the use of Robin boundary conditions leading to

$$\begin{aligned} (-\Delta - k^2)u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma^{\text{in}}, \\ \partial_n u &= i k u && \text{on } \Gamma^{\text{out}}. \end{aligned} \quad (2.5)$$

The weak formulation of this equation is given by: Find $u \in \mathcal{H}$ such that

$$A_{\text{Robin}}(u, v) := \int_{\Omega} (\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v}) - \int_{\Gamma^{\text{out}}} i k u \bar{v} = \int_{\Omega} f \bar{v} \quad \text{for all } v \in \mathcal{H}. \quad (2.6)$$

In most parts of this paper, we allow indeed that k is a function varying in Ω . In scaling $\text{diam}(\Omega^{\text{in}}) = 1$, the wavenumber k is dimensionless. The following conditions are always assumed to be satisfied:

$$\begin{aligned} k &\in L^\infty(\mathbb{R}^d, \mathbb{R}), \quad 0 \leq \text{essinf}_{x \in \Omega} k(x) \leq \text{esssup}_{x \in \Omega} k(x) =: k_{\max} < \infty, \\ k &= k_{\text{const}} > 1 \text{ outside a large ball,} \\ k &= k_{\text{const}} \text{ in a neighborhood } \mathcal{U}_{\text{const}}^* \text{ of } \Gamma^{\text{out}} \text{ and anywhere outside } \Omega^*. \end{aligned} \quad (2.7)$$

²For given data $g \in H^{1/2}(\Gamma^{\text{out}})$, the function $u_g \in H_{\text{loc}}^1(\mathbb{R}^d \setminus \overline{\Omega^*})$ is well defined as the weak solution of $(-\Delta - k^2)u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega^*}$ which satisfies Sommerfeld's radiation conditions as well as $u_g = g$ on Γ^{out} . Then $T_k g := \partial_n u_g$.

We expect that the condition $k > 1$ in (2.7) can be weakened to $k \geq 0$ for our a posteriori error estimates as long as the Dirichlet part Γ^{in} has positive measure. However, the low frequency case $0 \leq k \leq 1$ corresponds to the proper elliptic case and can be treated by using the well established theory as, e.g., developed in [16].

Let $\mathcal{U}_{\text{const}} := \mathcal{U}_{\text{const}}^* \cap \overline{\Omega}$. The constants in the estimates in this paper will depend on k_{max} , and $\mathcal{U}_{\text{const}}$ (through a trace inequality as in Lemma 2.6) but hold uniformly for all functions k satisfying (2.7).

2.2. Abstract Variational Formulation

Notation 2.1. All functions in this paper are in general complex-valued. For a Lebesgue measurable set $\omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$, $m \in \mathbb{N}$, we denote by $L^p(\omega)$ the usual Lebesgue space with norm $\|\cdot\|_{L^p(\omega)}$, scalar product $(\cdot, \cdot)_{L^2(\omega)}$ for $p = 2$, and by $H^m(\omega)$ the usual Sobolev spaces with norm $\|\cdot\|_{H^m(\omega)}$. The seminorm which contains only the derivatives of highest order is denoted by $|\cdot|_{H^m(\omega)}$. We equip the space \mathcal{H} , defined in (2.3), with the norm

$$\|v\|_{\mathcal{H};\Omega} := (\|\nabla v\|_{L^2(\Omega)}^2 + \|k_+ v\|_{L^2(\Omega)}^2)^{1/2} \quad \text{with } k_+ := \max\{1, k\}$$

which is obviously equivalent to the $H^1(\Omega)$ -norm.

Both sesquilinear forms A_{DtN} (2.4) and A_{Robin} (2.6) belong to the following class of forms (see Proposition 2.5).

Assumption 2.2 (Variational formulation). *Let $\Omega \subset \mathbb{R}^d$, for $d \in \{2, 3\}$, be a bounded Lipschitz domain. Then \mathcal{H} , equipped with the norm $\|\cdot\|_{\mathcal{H};\Omega}$, is a closed subspace of $H^1(\Omega)$. We consider a sesquilinear form $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ that can be decomposed into*

$$A = a - b,$$

where

$$a(v, w) := \int_{\Omega} (\langle \nabla v, \nabla \bar{w} \rangle - k^2 v \bar{w}) = (\nabla v, \nabla w)_{L^2(\Omega)} - (kv, kw)_{L^2(\Omega)}$$

and the sesquilinear form b satisfies the following properties:

(a) $b : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is a continuous sesquilinear form with

$$|b(v, w)| \leq C_b \|v\|_{\mathcal{H};\Omega} \|w\|_{\mathcal{H};\Omega} \quad \text{for all } v, w \in \mathcal{H},$$

for some positive constant C_b .

(b) There exist k -independent numbers $\theta \geq 0$ and $\gamma_{\text{ell}} > 0$ such that the following Gårding inequality holds:

$$\text{Re}(A(v, v)) + \theta \|k_+ v\|_{L^2(\Omega)}^2 \geq \gamma_{\text{ell}} \|v\|_{\mathcal{H};\Omega}^2 \quad \text{for all } v \in \mathcal{H}. \quad (2.8)$$

(c) The adjoint problem: Given $g \in L^2(\Omega)$, find $z \in \mathcal{H}$ such that

$$A(v, z) = (v, g)_{L^2(\Omega)} \quad \text{for all } v \in \mathcal{H} \quad (2.9)$$

is uniquely solvable and defines a bounded solution operator $Q_k^* : L^2(\Omega) \rightarrow \mathcal{H}$, $g \mapsto z$, more precisely, the (k -dependent) constant

$$C_k^{\text{adj}} := \sup_{g \in L^2(\Omega) \setminus \{0\}} \frac{\|Q_k^*(k_+^2 g)\|_{\mathcal{H};\Omega}}{\|k_+ g\|_{L^2(\Omega)}} \quad (2.10)$$

is finite.

Remark 2.3. Note that the different scaling of the constant C_k^{adj} with respect to k compared to the corresponding constant, say \tilde{C}_k^{adj} , in [14, (4.4)] and [15, (3.10)] becomes necessary since we allow for non-constant wavenumbers here and cannot take k as a simple multiplicative scaling factor out of the norms in (2.10). For constant k , the relation of these constants is $C_k^{\text{adj}} = k\tilde{C}_k^{\text{adj}}$.

Problem 2.4. Let A be a sesquilinear form as in Assumption 2.2. For given $f \in L^2(\Omega)$ we seek $u \in \mathcal{H}$ such that

$$A(u, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in \mathcal{H}. \quad (2.11)$$

Proposition 2.5. Both sesquilinear forms A_{Robin} (2.6) and A_{DtN} (2.4) (under the additional condition that Γ^{out} is a sufficiently large sphere) satisfy Assumption 2.2 with $\gamma_{\text{ell}} = 1$ and $\theta = 2$.

Proof. The proof is a slight modification of the corresponding proofs for constant wavenumber k in [14] and [12]. Condition (a) for A_{Robin} will follow from Lemma 2.6. For A_{DtN} we employ that k is constant in $\mathcal{U}_{\text{const}}$ and Γ^{out} is a sphere of a large radius $R > 0$. Hence, from the proof of [14, Lemma 3.3] it follows that

$$|(T_k u, v)_{\Gamma^{\text{out}}}| \leq C(R^{-1} \|u\|_{H^{1/2}(\Gamma^{\text{out}})} \|v\|_{H^{1/2}(\Gamma^{\text{out}})} + k_{\text{const}} \|u\|_{L^2(\Gamma^{\text{out}})} \|v\|_{L^2(\Gamma^{\text{out}})}).$$

By again using Lemma 2.6 we obtain

$$|(T_k u, v)_{\Gamma^{\text{out}}}| \leq C\left(1 + \frac{1}{R}\right) C_{\text{tr}}^2 \|u\|_{\mathcal{H};\Omega} \|v\|_{\mathcal{H};\Omega}$$

and the continuity of A_{DtN} follows.

For condition (b) and Robin boundary conditions, we employ

$$\text{Re}(A_{\text{Robin}}(v, v)) + 2\|k_+ v\|_{L^2(\Omega)}^2 \geq \int_{\Omega} (|\nabla v|^2 + k_+^2 |v|^2 + (k_+^2 - k^2) |v|^2) \geq \|v\|_{\mathcal{H};\Omega}^2$$

and (2.8) holds with $\theta = 2$ and $\gamma_{\text{ell}} = 1$. For the sesquilinear form A_{DtN} , we employ [14, Lemma 3.3 (2)] to obtain

$$\begin{aligned} \text{Re}(A_{\text{DtN}}(v, v)) + 2\|k_+ v\|_{L^2(\Omega)}^2 &\geq \left(\int_{\Omega} (|\nabla v|^2 + k_+^2 |v|^2 + (k_+^2 - k^2) |v|^2) - \text{Re}((T_k u, v)_{\Gamma^{\text{out}}}) \right) \\ &\geq \|v\|_{\mathcal{H};\Omega}^2 \end{aligned}$$

and (2.8) again holds with $\theta = 2$ and $\gamma_{\text{ell}} = 1$.

For condition (c), we may apply Fredholm's theory and, hence, it suffices to prove that

$$a(u, v) - b(u, v) = 0 \quad \text{for all } v \in \mathcal{H} \quad (2.12)$$

implies $u = 0$. For Robin boundary conditions, we argue as in [12, (8.1.2)] and for DtN boundary conditions as in the proof of [14, Theorem 3.8] to see that (2.12) implies $u \in H_0^1(\Omega)$. Hence, u solves

$$\int_{\Omega} (\langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v}) = 0 \quad \text{for all } v \in \mathcal{H}.$$

Let Ω^{**} be a bounded domain such that $\Omega \subset \Omega^{**} \subset \mathbb{R}^d \setminus \overline{\Omega^{\text{in}}}$ and $\Gamma^{\text{out}} \subset \Omega^{**}$. The extension of u by zero to Ω^{**} is denoted by u_0 . It satisfies $u \in \mathcal{H}(\Omega^{**}) := \{u \in H^1(\Omega^{**}) \mid u|_{\Gamma^{\text{in}}} = 0\}$ and

$$\int_{\Omega} (\langle \nabla u_0, \nabla \bar{v} \rangle - k^2 u_0 \bar{v}) = 0 \quad \text{for all } v \in \mathcal{H}(\Omega^{**}).$$

Elliptic regularity theory implies that $u_0 \in H^2(Q)$ for any compact subset $Q \subset \Omega^{**}$, in particular, in an open Ω^{**} neighborhood of Γ^{out} . The unique continuation principle (cf. [11, Chapter 4.3]) implies that $u_0 = 0$ in Ω^{**} so that $u = 0$ in Ω . \square

Lemma 2.6. *There exists a constant C_{tr} depending only on $\mathcal{U}_{\text{const}}$ such that*

$$\|v\|_{H^{1/2}(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|v\|_{\mathcal{H}; \mathcal{U}_{\text{const}}} \quad \text{for all } v \in H^1(\Omega)$$

and

$$\|v\|_{L^2(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|v\|_{L^2(\mathcal{U}_{\text{const}})}^{1/2} \|v\|_{H^1(\mathcal{U}_{\text{const}})}^{1/2} \quad \text{for all } v \in H^1(\Omega).$$

For k as in (2.7) we have

$$\|\sqrt{k}v\|_{L^2(\Gamma^{\text{out}})} \leq C_{\text{tr}} \|v\|_{\mathcal{H}; \mathcal{U}_{\text{const}}} \leq C_{\text{tr}} \|v\|_{\mathcal{H}; \Omega} \quad \text{for all } v \in H^1(\Omega).$$

Proof. The first two inequalities are standard Sobolev trace estimates [8, Lemma 3.1]. To prove the third inequality we note that due to $k = k_{\text{const}}$ on $\mathcal{U}_{\text{const}}$, there holds

$$\begin{aligned} k_{\text{const}} \|u\|_{L^2(\Gamma^{\text{out}})}^2 &\leq C_{\text{tr}}^2 k_{\text{const}} \|u\|_{L^2(\mathcal{U}_{\text{const}})} \|u\|_{H^1(\mathcal{U}_{\text{const}})} \\ &\leq \frac{C_{\text{tr}}^2}{2} (k_{\text{const}}^2 \|u\|_{L^2(\mathcal{U}_{\text{const}})}^2 + \|u\|_{H^1(\mathcal{U}_{\text{const}})}^2) \\ &= \frac{C_{\text{tr}}^2}{2} ((1 + k_{\text{const}}^2) \|u\|_{L^2(\mathcal{U}_{\text{const}})}^2 + |u|_{H^1(\mathcal{U}_{\text{const}})}^2) \\ &\leq C_{\text{tr}}^2 (\|k_+ u\|_{L^2(\mathcal{U}_{\text{const}})}^2 + |u|_{H^1(\mathcal{U}_{\text{const}})}^2). \end{aligned} \quad \square$$

2.3. Discretization

2.3.1. Conforming Galerkin Discretization. A conforming Galerkin discretization of Problem 2.4 is based on the definition of a finite dimensional subspace $S \subset \mathcal{H}$ and is given by: Find $u_S \in S$ such that

$$A(u_S, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in S. \quad (2.13)$$

2.3.2. hp-Finite Elements. As an example for S as above, we will define hp -finite elements on a finite element mesh \mathcal{T} consisting of simplices $K \in \mathcal{T}$ which are affine images of a *reference simplex*. More precisely, \mathcal{T} is a partition of Ω into d -dimensional disjoint open simplices which satisfy $\overline{\Omega} = \bigcup_{K \in \mathcal{T}} \overline{K}$ and, for any two non-identical elements $K, K' \in \mathcal{T}$, the intersection $\overline{K} \cap \overline{K'}$ either is empty, a common vertex (interval endpoint in 1-d), a common edge (for $d \geq 2$), or a common face (for $d = 3$). The local mesh width is given by the diameter h_K of the simplex K and p_K denotes the local polynomial degree on K .

Assumption 2.7. *Concerning the local mesh widths h_K and local polynomial degrees p_K , it will be convenient (cf. [16, (10)]) to assume that they are comparable to those of neighboring elements: There exist constants $c_{\text{shape}}, c_{\text{deg}} > 0$ such that*

$$\begin{aligned} c_{\text{shape}}^{-1} h_{K'} &\leq h_K \leq c_{\text{shape}} h_{K'}, \\ c_{\text{deg}}^{-1} p_{K'} &\leq p_K \leq c_{\text{deg}} p_{K'} \quad \text{for all } K, K' \in \mathcal{T} \text{ with } \overline{K} \cap \overline{K'} = \emptyset. \end{aligned} \quad (2.14)$$

The first requirement is also referred to as shape-regularity and is a measure for possible distortions of the elements. Many of the constants in the following estimates will depend on $c_{\text{shape}}, c_{\text{deg}}$.

Definition 2.8 (*hp*-finite element space). For meshes \mathcal{T} with element maps F_K as in Assumption 2.7 the *hp*-finite element space of piecewise (mapped) polynomials is given by

$$S^p(\mathcal{T}) := \{v \in \mathcal{H} : v|_K \circ F_K \in \mathbb{P}_{p_K} \text{ for all } K \in \mathcal{T}\},$$

where \mathbb{P}_q denotes the space of polynomials of degree $q \in \mathbb{N}$. For chosen \mathcal{T} and polynomial degree vector p , we may let $S = S^p(\mathcal{T})$.

3. A Priori Analysis

In this section, we collect those results on existence, uniqueness, stability, and regularity for the Helmholtz problem (2.4), which later will be used for the analysis of the a posteriori error estimator.

3.1. Well-Posedness

Proposition 3.1. *Let $\Omega^{\text{in}} \subset \mathbb{R}^d$, $d = 2, 3$, in (2.1a) be a bounded Lipschitz domain which is star-shaped with respect to the origin. Let $\Gamma^{\text{out}} := \partial B_R$ for some $R > 0$. Then (2.4) admits a unique solution $u \in \mathcal{H}$ for all $f \in \mathcal{H}'$ which depends continuously on the data.*

Proposition 3.2. *Let Ω be a bounded Lipschitz domain. For all $f \in (H^1(\Omega))'$, a unique solution u of problem (2.6) exists and depends continuously on the data.*

For the proofs of these propositions for constant k we refer, e.g., to [12, Proposition 8.1.3] and [7, Lemma 3.3], while for variable k one may argue as in Proposition 2.5.

3.2. Discrete Stability and Convergence

An essential role for the stability and convergence of the Galerkin discretization is played by the adjoint approximability which has been introduced in [15]; see also [6, 17]. (Note that the scaling with respect to the wavenumber differs from the scaling in the quoted papers for the same reasons as for Q_k^* as explained in Remark 2.3.)

Definition 3.3 (Adjoint approximability). For a finite dimensional subspace $S \subset \mathcal{H}$, we define the *adjoint approximability* of Problem 2.4 by

$$\sigma_k^*(S) := \sup_{g \in L^2(\Omega) \setminus \{0\}} \frac{\inf_{v \in S} \|Q_k^*(k_+^2 g) - v\|_{\mathcal{H}; \Omega}}{\|k_+ g\|_{L^2(\Omega)}}, \quad (3.1)$$

where Q_k^* is as in (2.10).

Theorem 3.4 (Stability and convergence). *Let $\gamma_{\text{ell}}, \theta, C_b, C_k^{\text{adj}}$ be as in Assumption 2.2 and S as in Section 2.3.1. Then the condition*

$$\sigma_k^*(S) \leq \frac{\gamma_{\text{ell}}}{2\theta(1 + C_b)} \quad (3.2)$$

implies the following statements:

(a) *The discrete inf-sup condition is satisfied:*

$$\inf_{v \in S \setminus \{0\}} \sup_{w \in S \setminus \{0\}} \frac{|A(v, w)|}{\|v\|_{\mathcal{H}; \Omega} \|w\|_{\mathcal{H}; \Omega}} \geq \frac{\gamma_{\text{ell}}}{2 + \gamma_{\text{ell}}/(1 + C_b) + 2\theta C_k^{\text{adj}}} > 0.$$

(b) *The Galerkin method based on S is quasi-optimal, i.e., for every $u \in \mathcal{H}$ there exists a unique $u_S \in S$ with $A(u - u_S, v) = 0$ for all $v \in S$, and there holds*

$$\begin{aligned} \|u - u_S\|_{\mathcal{H}; \Omega} &\leq \frac{2}{\gamma_{\text{ell}}} (1 + C_b) \inf_{v \in S} \|u - v\|_{\mathcal{H}; \Omega}, \\ \|k_+(u - u_S)\|_{L^2(\Omega)} &\leq \frac{2}{\gamma_{\text{ell}}} (1 + C_b)^2 \sigma_k^*(S) \inf_{v \in S} \|u - v\|_{\mathcal{H}; \Omega}. \end{aligned}$$

Proof. The proof follows very closely the proofs of [14, Theorems 4.2, 4.3] considering variable k . As an example we show the intermediate result of

$$\|k_+(u - u_S)\|_{L^2(\Omega)} \leq (1 + C_b) \|u - u_S\|_{\mathcal{H}; \Omega} \sigma_k^*(S).$$

For $e = u - u_S$ and $z = Q_k^*(k_+^2 e)$ it holds that

$$\begin{aligned} \|k_+ e\|_{L^2(\Omega)}^2 &= (k_+^2 e, e)_{L^2(\Omega)} = A(e, z) = A(e, z - z_S) \\ &\leq (1 + C_b) \|e\|_{\mathcal{H}; \Omega} \|z - z_S\|_{\mathcal{H}; \Omega} \leq (1 + C_b) \|e\|_{\mathcal{H}; \Omega} \sigma_k^*(S) \|k_+ e\|_{L^2(\Omega)}. \end{aligned} \quad \square$$

Remark 3.5. In [14, 15], it is proved for constant wavenumber k , that for S as in Section 2.3.2, i.e., hp -finite elements, the conditions

$$p = O(\log(k)) \quad \text{and} \quad \frac{kh}{p} = O(1) \quad (3.3)$$

imply (3.2), i.e.,

$$\sigma_k^*(S) \stackrel{\substack{\text{Proof of [14, Corollary 5.6] \\ \text{and [15, Proposition 5.3]}}}{\leq} C \left(1 + \frac{kh}{p}\right) \left(\frac{kh}{p} + k \left(\frac{kh}{\sigma p}\right)^p\right) \stackrel{(3.3)}{\leq} \frac{\gamma_{\text{ell}}}{2\theta(1 + C_b)},$$

where the constants in the $O(\cdot)$ terms in (3.3) depend only on the k -independent constants γ_{ell} , θ , C_b , and σ . This leads to the finite element space with only $O(k^d)$ degrees of freedom for a stable discretization of the Helmholtz equations. In this light, terms in the a posteriori error estimates which grow polynomially in p are expected to grow, at most, logarithmically with respect to k and, hence, are moderately bounded, also for large wavenumbers.

4. A Posteriori Error Estimation

The following assumption collects the requirements for the a posteriori error estimation.

Assumption 4.1. (a) *The continuous Helmholtz problem satisfies Assumption 2.2.*

(b) *S is a hp -finite element space as explained in Section 2.3.2 and satisfies Assumption 2.7.*

(c) *$u_S \in S$ is the computed solution satisfying the Galerkin equation.*

Remark 4.2. (a) Assumption 4.1 does not require the stability condition (3.2) to be satisfied which is only *sufficient* for existence and uniqueness of the discrete problem. We only assume that u_S exists, is computed, and solves the Galerkin equation for the specific problem. To be on the safe side in the case of constant wavenumber k , one can start the discretization process with the a priori choice (3.3) of p and h which implies (3.2) and, in turn, the existence and uniqueness of a Galerkin solution for any right-hand side in $L^2(\Omega)$.

(b) The constant in the a posteriori error estimate will contain the term $\sigma_k^*(S)$ as a factor. In order to get an explicit upper bound, an a priori estimate of the quantity is required which can be found for constant wavenumber in [14, Theorem 5.5] and [15, Propositions 5.3, 5.6] (cf. Remark 3.5).

For a simplicial finite element mesh \mathcal{T} , the boundary of any element $K \in \mathcal{T}$ consists of $(d-1)$ -dimensional simplices. We call (the relatively open interior of) these lower dimensional simplices the *edges* of K , although this terminology is related to the case $d = 2$. The set of all edges of all elements in \mathcal{T} is denoted by \mathcal{E}^* . The subset $\mathcal{E}^\partial \subset \mathcal{E}^*$ consists of all edges which are contained in Γ^{out} while the subset $\mathcal{E}^\Omega \subset \mathcal{E}^*$ consists of all edges that are contained in Ω . Finally, we set $\mathcal{E} := \mathcal{E}^\Omega \cup \mathcal{E}^\partial$, the set of all edges that are not in Γ^{in} . For a $K \in \mathcal{T}$ we define simplex neighborhoods about K by

$$\begin{aligned}\omega_K^{(0)} &:= \{\overline{K}\}, \\ \omega_K^{(j)} &:= \bigcup \{ \overline{K'} \mid K' \in \mathcal{T} \text{ and } \overline{K'} \cap \omega_K^{(j-1)} \neq \emptyset \} \quad \text{for } j \geq 1, \\ \mathcal{E}_K &:= \{E \in \mathcal{E} \mid E \subset \partial K\}.\end{aligned}\tag{4.1}$$

Definition 4.3 (Residual). For $v \in S$ we define the *volume residual* $\text{res}(v) \in L^2(\Omega)$ and the *edge residual* $\text{Res}(v) \in L^2(\bigcup_{E \in \mathcal{E}} E)$ by

$$\begin{aligned}\text{res}(v) &:= f + \Delta v + k^2 v \quad \text{on } K \in \mathcal{T}, \\ \text{Res}(v) &:= \begin{cases} [\partial_n v]_E & \text{on } E \in \mathcal{E}^\Omega, \\ -\partial_n v + i k v & \text{on } E \in \mathcal{E}^\partial. \end{cases}\end{aligned}$$

Here $[v]_E$ is the *jump* of the given function v on the edge E , i.e., the difference of the limits in points $x \in E$ from both sides.

In the definitions above we used exact data f, k . We will later, Section 4.2, replace these by approximations.

The residual $\text{Res}(v)$ is defined for the Robin boundary condition (2.5) for simplicity. With an obvious modification of this definition, we could also insert a term $T_k v$ here, instead of $(i k v)$, for the DtN boundary condition (2.2).

Definition 4.4 (Error estimator). We define for $v \in S$ the *error indicator*

$$\eta(v) := \left(\sum_{K \in \mathcal{T}} \alpha_K^2 \|\text{res}(v)\|_{L^2(K)}^2 + \sum_{E \in \mathcal{E}} \alpha_E^2 \|\text{Res}(v)\|_{L^2(E)}^2 \right)^{1/2}, \tag{4.2}$$

where the weights $\{\alpha_K, \alpha_E \mid K \in \mathcal{T}, E \in \mathcal{E}\}$ are given by

$$\alpha_K := \frac{h_K}{p_K}, \quad \alpha_E := \left(\frac{h_E}{p_E} \right)^{1/2}.$$

The choice of the weights α_K , α_E in (4.2) is related to an interpolation estimate in [16, Theorems 2.1, 2.2] which we explain next.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^2$ and let $p = (p_K)_{K \in \mathcal{T}}$ denote a polynomial degree distribution satisfying (2.14). Let Assumption 4.1 (b) be satisfied. Then there exists a constant $C_0 > 0$, that depends only on the shape-regularity of the mesh (cf. Assumption 2.7), and a bounded linear operator $I_S : H_{\text{loc}}^1(\mathbb{R}^2) \rightarrow S$ such that for all simplices $K \in \mathcal{T}$, all edges $E \in \mathcal{E}_K$, and $v \in \mathcal{H}$ we have*

$$\frac{p_K}{h_K} \|v - I_S v\|_{L^2(K)} + \sqrt{\frac{p_E}{h_E}} \|v - I_S v\|_{L^2(E)} \leq C_0 \|\nabla v\|_{L^2(\omega_K^{(4)})}. \quad (4.3)$$

Proof. This result has been proven in [16] in a vertex oriented setting, but is easily reformulated as stated above using shape uniformity and quasi-uniformity in the polynomial degree. \square

Theorem 4.8 will show that $\eta(u_S)$ can be used for a posteriori error estimation. That it estimates the error from above is called *reliability*, that it estimates the error from below is called *efficiency*.

4.1. Reliability

According to Assumption 4.1 the exact solution $u \in \mathcal{H}$ and the Galerkin solution $u_S \in S$ of (2.11) and (2.13), respectively, exist. In view of inequality (2.8), we estimate the error $e = u - u_S$, $\text{Re}(A(e, e))$, and $\|k_+ e\|_{L^2(\Omega)}$ separately in terms of $\eta(u_S)$.

Lemma 4.6. *Let Assumption 4.1 be satisfied. Then there is a constant $C_1 > 0$, that depends only on the shape-regularity of the mesh (cf. Assumption 2.7), such that*

$$|\text{Re}(A(e, e))| \leq C_1 \eta(u_S) \|e\|_{\mathcal{H}; \Omega}.$$

Proof. We start with an auxiliary computation. Let $e = u - u_S$ denote the Galerkin error and let $w \in \mathcal{H}$. By using solution properties, e.g., the Galerkin orthogonality and integration by parts, we obtain the error representation

$$\begin{aligned} A(e, w) &= A(e, w - I_S w) \\ &= \sum_{K \in \mathcal{T}} (\text{res}(u_S), w - I_S w)_{L^2(K)} + \sum_{E \in \mathcal{E}} (\text{Res}(u_S), w - I_S w)_{L^2(E)}. \end{aligned}$$

We use the assumed interpolation estimates (4.3) and get with the Cauchy–Schwarz inequality, applied to integrals and sums,

$$\begin{aligned} |A(e, w)| &\leq C_0 \left(\sum_{K \in \mathcal{T}} \alpha_K^2 \|\text{res}(u_S)\|_{L^2(K)}^2 + \sum_{E \in \mathcal{E}} \alpha_E^2 \|\text{Res}(u_S)\|_{L^2(E)}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \|w\|_{\mathcal{H}; \omega_K^{(4)}}^2 \right)^{1/2} \\ &\leq C_1 \eta(u_S) \|w\|_{\mathcal{H}; \Omega}. \end{aligned} \quad (4.4)$$

By setting $w = e$ and using $|\text{Re}(A(e, w))| \leq |A(e, w)|$ we obtain the assertion. \square

Lemma 4.7. *Let Assumption 4.1 be satisfied. Then, with C_1 from Lemma 4.6 and $\sigma_k^*(S)$ as in (3.1),*

$$\|k_+ e\|_{L^2(\Omega)} \leq C_1 \sigma_k^*(S) \eta(u_S). \quad (4.5)$$

Proof. We define z by (2.9) with $g := k_+^2 e$. Let $z_S \in S$ denote the best approximation of z with respect to the $\|\cdot\|_{\mathcal{H};\Omega}$ -norm. By using (4.4) with $w = z - z_S$ and the definition of the adjoint approximation property, we obtain

$$\begin{aligned} \|k_+ e\|_{L^2(\Omega)}^2 &= A(e, z) = A(e, z - z_S) \leq C_1 \eta(u_S) \|z - z_S\|_{\mathcal{H};\Omega} \\ &\leq C_1 \eta(u_S) \sigma_k^*(S) \|k_+ g\|_{L^2(\Omega)} \end{aligned}$$

and this gives (4.5). \square

Theorem 4.8 (Reliability estimate). *Let Assumption 4.1 be satisfied. Then, with C_1 from Lemma 4.6,*

$$\|e\|_{\mathcal{H};\Omega} \leq \frac{1}{\gamma_{\text{ell}}} C_1 (1 + (\gamma_{\text{ell}} \theta)^{1/2} \sigma_k^*(S)) \eta(u_S).$$

Proof. The combination of (2.8), (4.5) with the bounds obtained in Lemmata 4.6 and 4.7 yields

$$\gamma_{\text{ell}} \|e\|_{\mathcal{H};\Omega}^2 \leq \text{Re}(A(e, e)) + \theta \|k_+ e\|_{L^2(\Omega)}^2 \leq C_1 \eta(u_S) \|e\|_{\mathcal{H};\Omega} + \theta C_1^2 \sigma_k^*(S)^2 \eta(u_S)^2$$

so that

$$\begin{aligned} \|e\|_{\mathcal{H};\Omega} &\leq \frac{1}{\gamma_{\text{ell}}} C_1 \eta(u_S) + \left(\frac{\theta}{\gamma_{\text{ell}}} \right)^{1/2} C_1 \sigma_k^*(S) \eta(u_S) \\ &\leq \frac{1}{\gamma_{\text{ell}}} C_1 (1 + (\gamma_{\text{ell}} \theta)^{1/2} \sigma_k^*(S)) \eta(u_S). \end{aligned} \quad \square$$

In the previous arguments res and Res were defined with exact data functions f, k . In order to obtain *two-sided* bounds for the Galerkin error in terms of an error estimator one has to modify the error estimator by data oscillations: For any $K \in \mathcal{T}$ we define the local and global data oscillations by

$$\text{osc}_K^2 := \alpha_K^2 (\|f - \tilde{f}\|_{L^2(\omega_K)}^2 + \|(k^2 - \tilde{k}^2)u_S\|_{L^2(\omega_K)}^2) \quad \text{and} \quad \text{osc} := \sqrt{\sum_{K \in \mathcal{T}} \text{osc}_K^2}.$$

The modified error estimator $\tilde{\eta}(u_S)$ is defined by replacing f and k in the definitions of res and Res in (4.3) by local $L^2(K)$ -projections \tilde{f}, \tilde{k} of degree p_K or some degree $q_K \sim p_K$.

Corollary 4.9. *Let Assumption 4.1 be satisfied. Then the modified error estimate is reliable up to data oscillations:*

$$\|e\|_{\mathcal{H};\Omega} \leq \frac{\sqrt{3}}{\gamma_{\text{ell}}} C_1 (1 + (\gamma_{\text{ell}} \theta)^{1/2} \sigma_k^*(S)) (\tilde{\eta}(u_S) + \text{osc}).$$

Proof. We notice that

$$\begin{aligned} \text{res}(u_S) &= f + k^2 u_S + \Delta u_S = \tilde{f} + \tilde{k}^2 u_S + \Delta u_S + f - \tilde{f} + (k^2 - \tilde{k}^2) u_S \\ &= \widetilde{\text{res}}(u_S) + f - \tilde{f} + (k^2 - \tilde{k}^2) u_S \end{aligned}$$

and, on Γ^{out} ,

$$\text{Res}(u_S) = -\partial_n u_S + i k u_S = \widetilde{\text{Res}}(u_S),$$

since k is constant on Γ^{out} . The same argument applies to $T_k u_S$. We thus obtain

$$\eta(u_S)^2 \leq 3 \tilde{\eta}(u_S)^2 + 3 \sum_{K \in \mathcal{T}} \alpha_K^2 \|f - \tilde{f}\|_{L^2(K)}^2 + 3 \sum_{K \in \mathcal{T}} \alpha_K^2 \|(k^2 - \tilde{k}^2)u_S\|_{L^2(K)}^2.$$

Hence, the assertion follows from Theorem 4.8. \square

An explicit estimate of the error by the error estimator requires an upper bound for the adjoint approximation property $\sigma_k^*(S)$. Such estimates for hp -finite elements spaces for constant wavenumbers k are derived in [14] and [15] for problem (2.4) and (2.6), respectively. We summarize the results as the following corollaries.

Corollary 4.10 (Robin boundary conditions). *Consider problem (2.6) with constant wavenumber k , where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded Lipschitz domain. Either Ω has an analytic boundary or it is a convex polygon in \mathbb{R}^2 with vertices A_j , $j = 1, \dots, J$. We use the approximation space S described in Section 2.3.2. If Ω is a polygon, then the hp -finite element space S is employed where, in addition, $L = O(p)$ geometric mesh grading steps are performed towards the vertices – for the details we refer to [15].*

Let $f \in L^2(\Omega)$ and $k > 1$ and assume that $\Gamma^{\text{in}} = \emptyset$, i.e., we consider the pure Robin problem. Let Assumption 4.1 (a) and (b) be satisfied.

Then there exist constants $\delta, \tilde{c} > 0$ that are independent of h, p , and k such that the conditions

$$\frac{kh}{p} \leq \delta \quad \text{and} \quad p \geq 1 + \tilde{c} \log(k)$$

imply (3.1) and thus the k -independent a posteriori error estimate

$$\|e\|_{\mathcal{H};\Omega} \leq C_1(1 + \sqrt{2}\tilde{C})\eta(u_S),$$

where \tilde{C} only depends on δ and \tilde{c} .

Corollary 4.11 (DtN boundary conditions). *Consider problem (2.4) for constant wave-number k , where Ω has an analytic boundary. Let Assumption 4.1 (a) and (b) be satisfied and assume that the constant C_k^{adj} in (2.10) grows at most polynomially in k , i.e., there exists some $\beta \geq 0$ such that³ $C_k^{\text{adj}} \leq Ck^\beta$. Let $f \in L^2(\Omega)$ and $k > 1$.*

Then there exist constants $\delta, \tilde{c} > 0$ that are independent of h, p , and k such that the conditions

$$\frac{kh}{p} \leq \delta \quad \text{and} \quad p \geq 1 + \tilde{c} \log(k)$$

imply (3.1) and thus the k -independent a posteriori error estimate

$$\|e\|_{\mathcal{H};\Omega} \leq C_1(1 + \sqrt{2}\tilde{C})\eta(u_S),$$

where \tilde{C} only depends on δ and \tilde{c} .

4.2. Efficiency

The reliability estimate in the form of Theorem 4.8 allows us to use the error estimator as a stopping criterion in order to satisfy a certain given accuracy requirement. In order to achieve *efficiently* this goal one has to ensure that the true error is not overestimated too much by the error estimator and that its local spatial distribution is well represented. This is called *efficiency*. It is verified by estimating the local error by the localized error estimator. In this light, we define the localized version of the error estimator by

$$\eta_K(v) := \left(\alpha_K^2 \|\text{res}(v)\|_{L^2(K)}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K} \alpha_E^2 \|\text{Res}(v)\|_{L^2(E)}^2 \right)^{1/2},$$

with \mathcal{E}_K as in (4.1). Note that $\eta(v) = \sqrt{\sum_{K \in \mathcal{T}} \eta_K^2(v)}$.

³See [9] for sufficient conditions on the domain which implies this growth condition.

As in Corollary 4.9 let us define approximations \tilde{f}, \tilde{k} to f, k , respectively, as local $L^2(K)$ -projections onto a polynomial of degree p_K (or some $q_K \sim p_K$). In this case, we use the notation $\widetilde{\text{res}}$ and $\tilde{\eta}$ accordingly. Also we set

$$k_{K,+} := \max\{\|k\|_{L^\infty(K)}, 1\}$$

Theorem 4.12. *Let Assumption 4.1 and (2.7) be satisfied and let the mesh be shape regular (cf. Assumption 2.7). We assume that Ω is either an interval ($d = 1$), or a polygonal domain ($d = 2$), or a Lipschitz polyhedron ($d = 3$), and that the element maps F_K are affine. We assume the resolution condition:*

$$\frac{k_{K,+} h_K}{p_K} \lesssim 1 \quad \text{for all } K \in \mathcal{T}. \quad (4.6)$$

Then there exists a constant C depending only on the constants in Assumptions 2.7 and 2.2 – and which, in particular, is independent of k , p_K , h_K and u , u_S – so that

$$\tilde{\eta}_K(u_S) \leq C p_K^{3/2} (\|u - u_S\|_{\mathcal{H}; \omega_K} + \text{osc}_K). \quad (4.7)$$

Proof. We apply the results [16, Lemmata 3.4, 3.5]. There, the proofs are given for two space dimensions, i.e., $d = 2$. They carry over to the case $d = 1$ simply by using [16, Lemma 2.4] instead of [16, Theorem 2.5]. For the case $d = 3$, a careful inspection of the proofs of [16, Theorem 2.5] (which is given in [13, Theorem D2]) and [16, Lemma 2.6] shows that these lemmata also hold for $d = 3$. Hence, the proofs of [16, Lemmata 3.4, 3.5] can be used verbatim for the cases $d = 1$ and $d = 3$.

We choose $\alpha = 0$ in [16, Lemmata 3.4, 3.5]. Following these lines of arguments, we get for any $\varepsilon > 0$, $K \in \mathcal{T}$, $E \in \mathcal{E}_K$,

$$\begin{aligned} \alpha_K^2 \|\widetilde{\text{res}}(u_S)\|_{L^2(K)}^2 &\leq C(\varepsilon) (p_K^2 \|\nabla(u - u_S)\|_{L^2(K)}^2 \\ &\quad + p_K^{1+2\varepsilon} (\alpha_K^2 \|k^2(u - u_S)\|_{L^2(K)}^2 + \text{osc}_K^2)) \end{aligned}$$

and

$$\begin{aligned} \alpha_K \|\widetilde{\text{Res}}(u_S)\|_{L^2(E)}^2 &\leq C(\varepsilon) p_K^{2\varepsilon} (p_K^2 \|\nabla(u - u_S)\|_{L^2(\omega_K)}^2 \\ &\quad + p_K^{1+2\varepsilon} (\alpha_K^2 \|k^2(u - u_S)\|_{L^2(\omega_K)}^2 + \text{osc}_K^2)). \end{aligned}$$

Hence, since local diameters and polynomial degrees are comparable, we have

$$\begin{aligned} &\alpha_K^2 \|\widetilde{\text{res}}(u_S)\|_{L^2(K)}^2 + \alpha_E^2 \|\widetilde{\text{Res}}(u_S)\|_{L^2(E)}^2 \\ &\leq \alpha_K^2 \|\widetilde{\text{res}}(u_S)\|_{L^2(K)}^2 + C \alpha_K \|\widetilde{\text{Res}}(u_S)\|_{L^2(E)}^2 \\ &\leq C(\varepsilon) p_K^2 (1 + p_K^{2\varepsilon}) \\ &\quad \cdot (\|\nabla(u - u_S)\|_{L^2(\omega_K)}^2 + 4 p_K^{2\varepsilon-1} k_{K,+}^2 \alpha_K^2 \|k_+(u - u_S)\|_{L^2(\omega_K)}^2 + p_K^{2\varepsilon-1} \text{osc}_K^2). \end{aligned} \quad (4.8)$$

For the special choice $\varepsilon = 1/2$ and with condition (4.6) we finally get

$$\tilde{\eta}_K^2(u_S) \leq C p_K^3 (\|u - u_S\|_{\mathcal{H}; \omega_K}^2 + \text{osc}_K^2). \quad \square$$

Remark 4.13. (a) It is possible to choose any $\varepsilon > 0$ in (4.8) (with $C(\varepsilon) \sim 1/\varepsilon$). The factor $p_K^{3/2}$ in the estimate (4.7) then can be replaced by $p_K^{1+\varepsilon}$, while condition (4.6) has the weaker form $k_{K,+} h_K / p_K \leq p_K^{1/2-\varepsilon}$ (for $\varepsilon \leq 1/2$). However, in view of $p_K \sim \log(k)$ we think that this is of minor importance.

(b) Theorem 4.12 could be completed by a data saturation condition for the terms $p_K^{3/2} \text{osc}_K$ in (4.7), which would then allow to bound $\tilde{\eta}_K(u_S)$ directly by the error.

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